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Algebra 1

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Final

The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can't make sense of it in finite time you could lose coherent narrative through line. If he can't make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics.

Problem 1 :
Let G be a finite group of order $|G| = p^2$ for some prime $p > 1$.

1. **(3pts) Explain why there must exist cyclic subgroups of order p in G.**
2. **(3pts) Explain why all such subgroups must be normal in G.**
3. **(6pts) Deduce from the previous question, prove that $G \simeq \mathbb{Z}/p^2\mathbb{Z}$ or $G \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.**
   Hint : The Sylow theorems will not help. (Why ?).

Problem 2 :
Let $n = p^k$ for some prime $p > 1$ and $k \geqslant 1$. If $m \in \mathbb{N}$,

1. **(4pts) Prove that $gcd(m, p^k) \neq 1 \Leftrightarrow p$ *divides* m Thus**
$$U_{p^k} = \{[m] : 0 \leqslant m \leqslant p^k \textit{ and } m \textbf{ is not a multiple of } p\}$$
2. **(5pts) Prove that the group of units in $\mathbb{Z}/p^k\mathbb{Z}$ has order**
$$|U_{p^k}| = p^{k-1}(p-1) \textbf{ for all } k \geqslant 1$$
3. **(5pts) Deduce from the previous question the cardinality $|U_{37}|$ and $|U_{120}|$. (Hint : This can be answers without exhaustively listing the element in U_n.)**

Problem 3 :
Let $G = (U_{11}, \cdot)$ be the multiplicative group of units in $\mathbb{Z}/11\mathbb{Z}$. Make a table in which you list the orders of all elements $x \in U_{11}$ and the subgroups $H = < x >$ they generate. Use this to answer the following questions.

1. **(3pts) How many distinct cyclic subgroups are there in G ? Is G itself cyclic ?**

2. **(3pts) How many distinct 5-Sylow subgroups are there in G ? Describe them as subsets of G. Do the same for the 2-Sylow subgroups.**
3. **(4pts) When $n = 5$, how many distinct homomorphisms**

$$\Phi : (\mathbb{Z}/5\mathbb{Z}, +) \to (U_{11}, \cdot)$$

are there ? Describe them by telling me where in U_{11} each homomorphism $\Phi^{(i)}$ sends the additive generator $a = [1]_5$ of the group $(\mathbb{Z}/5\mathbb{Z}, +)$.

4. **(4pts) For $2 \leqslant n \leqslant 10$ fill in the values for $|U_n|$ in the following table.**

n	2	3	4	5	6	7	8	9	10
$\lvert U_n \rvert$									
Nontrivial Φ?									

Identify all such n for which there is a nontrivial homomorphism $\Phi : (\mathbb{Z}/3\mathbb{Z}, +) \to (U_n, \cdot)$. Explain.
Note : Φ is trivial if $\Phi(x)$ equals the identity element $e = [1]_n$ in U_n for all $x \in \mathbb{Z}/3\mathbb{Z}$. As a check on your calculations remember : $o(x)$ must divide $|U_{11}|$ for each $x \in U_{11}$.

Problem 4 : Short answer questions

1. **(4pts) List all elements of $(\mathbb{Z}/15\mathbb{Z}, +)$ that are additive generators of this group.**

2. **(8pts) Here is a list of several abelian group of order $|G| = 225 = 3^2 5^2$. Make a table identifying all pairs that are isomorphic. Which of these groups are cyclic ?**

G_1 $\mathbb{Z}/225\mathbb{Z}$
G_2 $\mathbb{Z}/25\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$
G_3 $\mathbb{Z}/25\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
G_4 $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$
G_5 $\mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
G_6 $\mathbb{Z}/75\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
G_7 $\mathbb{Z}/45\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$

Hint : For each G_i what is $max\{o(x)\}$.

3. **(8pts) In the dihedral group $D_n = \{\rho^i \sigma^j : i \in \mathbb{Z}/n\mathbb{Z}, j \in \mathbb{Z}/2\mathbb{Z}\}$ the normal subgroup $N =< \rho >$ has just 2 cosets, namely N and $N\sigma$.**

(a) Prove that every element of the "outside" coset $N\sigma$ has order 2.

(b) In the particular group D_7 exhibit examples of

i. A 7-Sylow subgroup ;

ii. A 2-Sylow subgroup ;

described as explicit subsets of $D_7 = \{\rho^i \sigma^j : i \in \mathbb{Z}/7\mathbb{Z}, j \in \mathbb{Z}/2\mathbb{Z}\}$.

4. (4pts) Given the following permutations factored as disjoint cycles, identify (if any) lie in the same conjugacy class in S_8.

(a) $C_1 = (1,2,3)(4857)$;

(b) $C_2 = (231)(5748)(6)$;

(c) $C_3 = (4382)(615)$;

(d) $C_4 = (12)(43)(685)$;

5. (4pt) If G is a finite group with $n = |G|$, explain why must have $x^n = e$ for all $x \in G$.

6. (10pts) Below we show the action of a certain permutation $\sigma \in S_{10}$ on the integers 1, 2, $\cdot$, 10.

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 5 & 9 & 8 & 10 & 3 & 2 & 4 & 6 & 7 \end{matrix}$$

Based on this information :

(a) Determine the decomposition of σ into disjoint cycles.

(b) Decide whether σ is odd or even.

(c) What is the order of this element in S_{10} ?

(d) What is the size of the conjugacy class of σ in S_{10} ?

(e) Determine the cycle decomposition of the conjugate $(135)\sigma(135)^{-1}$.

7. (6pts) Please note that you can use any question before even if you have not proven them.

(a) Show that

$$(Aut(\mathbb{Z}/11\mathbb{Z}), \circ) \cong (\mathbb{Z}/10\mathbb{Z}, \cdot)$$

(b) **Show that every homomorphism**

$$\phi : \mathbb{Z}/3\mathbb{Z} \to Aut(\mathbb{Z}/11\mathbb{Z})$$

is trivial, i.e $\phi(g) = Id$ **for every** $g \in \mathbb{Z}/3\mathbb{Z}$**.**

(c) **Show that every semi-direct product of** $\mathbb{Z}/11\mathbb{Z}$ **by** $\mathbb{Z}/3\mathbb{Z}$ **(with** $\mathbb{Z}/11\mathbb{Z}$ **being normal in the semi-direct product) is actually a direct product.**

8. **(6 pts)**

(a) **Describe the action of** G **on itself by conjugation and show it is an action.**

(b) **Show that under this action** $|orb(g)| = 1$ **if and only if** $g \in Z(G)$**.**

(c) **Suppose that** $|G| = p^n$**.**

i. **Using the properties of group actions to prove that if** $|orb(g)| \neq 1$ **then** $p \mid |orb(g)|$ **.**

ii. **Deduce from the previous question that** G **has non-trivial center.**